UNIT 14 : CONVERGENCE OF IMPROPER INTEGRALS

UNIT STRUCTURE

14.1 Learning Objectives

14.2 Introduction

14.3 Convergence of Improper Integrals of the First Kind

14.4 Tests for Convergence of Improper Integrals of the First Kind

14.5 Absolute Convergence

14.6 Convergence of Improper Integrals of the Second Kind

14.7 Tests for Convergence of Improper Integrals of the Second Kind

14.8 Let Us Sum Up

14.9 Answers to Check Your Progress

14.10 Further Reading

14.11 Model Questions

14.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- know about the convergence of improper integrals
- learn about the various test for convergence of improper integrals of the first kind
- know about absolute convergence
- learn about the tests for convergence of improper integrals of the second kind.

14.2 INTRODUCTION

In the previous unit on Riemann Integral, the theory of integration was developed under two assumptions.

(i) the interval of integration was required to be a closed and bounded interval, and

(ii) the integrand was required to be bounded on the interval.

If these restrictions are relaxed on the theory of integration, we

have the followed two types of integrals, called improper integrals or infinite integrals.

(i) improper integrals on a finite interval where the integrand is unbounded.

(ii) improper integrals on an unbounded interval.

Thus

$$\int_0^1 \frac{dx}{x^3}, \int_1^2 \frac{dx}{(1-x)(2-x)}, \int_{-\infty}^{\infty} \frac{dx}{1+x^2}, \int_0^{\infty} \frac{1}{\sqrt{x}} dx$$

are few examples of improper integrals where the restriction on the interval of integration is removed and the integrand is not bounded.

Based on the relaxation on the above conditions, we can classify the improper integral as follows :

Improper integral of the first kind : An improper integral with finite interval of integration [a, b] with its integrand having a finite number of points of infinite discontinuity

Improper integral of the second kind : An improper integral with infinite interval of integration and bounded integrand.

Improper integral of the mixed kind : An improper integral with infinite interval of integration with its integrand having a finite number of points of infinite discontinuity.

We define convergence of improper integrals of the first and second kind and discuss the properties of each type separately in the succeeding sections.

14.3 CONVERGENCE OF IMPROPER INTEGRALS OF THE FIRST KIND

Case I. Convergence of the improper integral $\int_a^b f(x)\,dx$ when a is the only point of infinite discontinuity of f in [a, b].

Let te left end point a of the closed and bounded interval [a, b] be the only point of infinite discontinuity of a function f whcih is bounded and integrable on $[a+\in, b]$ for every $\in$ satisfying $0 < \in < b-a$

Let

$$\phi(\in) = \int_{a+c}^{b} f(x)\,dx, \quad 0 < \in < b-a$$

If $\lim_{\in\to 0+}\phi(\in)$ exists (finitely) then the improper integral $\int_a^b f(x)\,dx$ is said to be convergent and we write $\int_a^b f(x)\,dx = l$ where l is the limit

If $\lim_{\in\to 0+}\phi(\in)$ does not exist (finitely) then the improper integral $\int_a^b f(x)\,dx$ is a said to be divergent.

Note : If a be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a+\in, b]$ for every $\in$ satisfying $0<\in<b-a$ and $\int_a^b f(x)\,dx$ is convergent, then $\int_a^c f(x)\,dx$ is also convergent for all $c\in(a,b)$

Example : The integral $\int_0^1 \frac{1}{x}dx$ is improper, since 0 is a point of infinite discontinuity of the integrand.

Solution : The integrand is bounded and integrable on $[0+\in, 1]$ for all $\in$ satisfying $0<\in<1$. Now,

$$\lim_{\in\to 0+}\int_{0+\in}^1 \frac{1}{x}dx = \lim_{\in\to 0+}\left[-\log\in\right] = \infty$$

Therefore the improper integral $\int_0^1 \frac{1}{x}dx$ is divergent.

Case II. Convergence of the improper integral $\int_a^b f(x)\,dx$ when b is the only point of infinite discontinuity of f in [a, b].

Let the right end point b of the closed and bounded interval [a, b] be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a, b-\in]$ for every $\in$ satisfying $0<\in<b-a$.

Let

$$\phi(\in) = \int_a^{b-c} f(x)\,dx, \quad 0<\in<b-a$$

If $\lim_{\in\to 0+}\phi(\in)$ exists (finitely then the improper integral $\int_a^b f(x)\,dx$ is a said to be convergent. If the limit be *l*, we write $\int_a^b f(x)\,dx = l$.

If $\lim_{\in\to 0+}\phi(\in)$ does not exist (finitely) then the improper integral $\int_a^b f(x)\,dx$ is a said to be divergent.

Note : If b be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a, b-\in]$ for every $\in$ satisfying $0<\in<b-a$

and $\int_a^b f(x)\,dx$ is convergent, then $\int_c^b f(x)\,dx$ is also convergent of $c \in (a, b)$.

Example : The integral $\int_0^1 \frac{1}{1-x}dx$ is improper, since 1 is a point of infinite discontinuity of the integrand.

Solution : The integrand is bounded and integrable on $[0, 1-\in]$ for all $\in$ satisfying $0 < \in < 1$. Now,

$$\lim_{\in \to 0+} \int_0^{1-\in} \frac{1}{1-x}dx = \lim_{\in \to 0+} \left[-\log \in\right] = \infty$$

Therefore the improper integral $\int_0^1 \frac{1}{1-x}dx$ is divergent.

Example : The integral $\int_0^1 \frac{1}{1-x^2}dx$ is improper, since 1 is a point of infinite discontinuity of the integrand.

Solution : The integrand is bounded and integrable on $[0, 1-\in]$ for all $\in$ satisfying $0 < \in < 1$. Now,

$$\lim_{\in \to 0+} \int_0^{1-\in} \frac{1}{\sqrt{1-x^2}}dx = \lim_{\in \to 0+} \left[\sin^{-1}(1-\in)\right] = \frac{\pi}{2}$$

Therefore the improper integral $\int_0^1 \frac{1}{1-x^2}dx$ is convergent of $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{2}$.

Case III. Convergence of the improper integral $\int_a^b f(x)\,dx$ when a and b is the only point of infinite discontinuity of f in [a, b]

Let the end points a, b of the closed and bounded interval [a, b] be the only points of infinite disontinuity of a function f which is bounded and ingegrable on $[a+\in, b-\in']$ for every $\in, \in'$ satisfying $0 < \in < b-a,\ 0 < \in' < b-a$.

Let $c \in (a, b)$

If the improper ingegrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ be both convergent according to the definitions given above, then the improper integral $\int_a^b f(x)\,dx$ is said to be convergent.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Note : If the end points a and be of the closed and bounded interval [a, b] be the only points of infinite discontinuity of a function f and the improper integral $\int_a^b f(x)\,dx$ be convergent, then for any point $d \in (a,b)$.

$$\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx$$

Example : The integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ is improper, since 0 and 2 are points of infinite discontinuity of the integrand.

Solution : The integrand is bounded and integrable on $[0+\in,\ 2-\in]$ for all $\in, \in'$ satisfying $0 < \in < 2,\ 0 < \in' < 2$.

Let us examine if $\lim_{\in\to 0+} \int_{0+\in}^1 \frac{1}{\sqrt{x(2-x)}}dx$ and

$\lim_{\in'\to 0+} \int_1^{2-\in'} \frac{1}{\sqrt{x(2-x)}}dx$ exist.

$$\lim_{\in\to 0+} \int_{0+\in}^1 \frac{1}{\sqrt{x(2-x)}}dx = \lim_{\in\to 0+} [x-1]\frac{1}{\in} = \frac{\pi}{2}$$

$$\lim_{\in'\to 0+} \int_1^{2-\in'} \frac{1}{\sqrt{x(2-x)}}dx = \lim_{\in'\to 0+} \left[\sin^{-1}(x-1)\right]_1^{2-\in'} = \frac{\pi}{2}$$

Therefore $\int_0^2 \frac{1}{\sqrt{x(2-x)}}dx$ is convergent and $\int_0^2 \frac{1}{\sqrt{x(2-x)}}dx = \pi$

Case IV. Convergence of the improper integral $\int_a^b f(x)\,dx$ when a finite number of points c_1, c_2, ...c_n, and finite number of points of discontinuity of f in [a, b], where $a \le c_1 \le c_2 < ..... < c_{n-1} < c_n \le b$.

If the improper integrals

$$\int_a^{c_1} f(x)dx,\ \int_{c_1}^{c_2} f(x)dx,\ \int_{c_2}^{c_3} f(x)dx, ... \int_{c_{n-1}}^{c_n} f(x)dx,\ \int_{c_n}^b f(x)dx$$

all exist in accordance with the definitions given above and we regard

$$\int_a^{c_1} f(x)dx = 0 \ \text{ if } \ c_1 = a \qquad \int_{c_n}^b f(x)dx = 0 \ \text{ if } \ b = c_n$$

then we say that $\int_a^b f(x)\,dx$ exists and write

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + ...\int_{c_n}^b f(x)dx$$



CHECK YOUR PROGRESS 1

Q 1: Show that the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent .

Q 2: Examine the convergence of the improper integrals :

a) $\int_0^1 \frac{1}{x^2}dx,$ b) $\int_0^{\pi/2} \sin x \log \sin x\ dx,$

c) $\int_0^1 \frac{1}{\sqrt{x-x^2}}dx$

Q 3: Show that if $\int_a^b f(x)\,dx$ converges at *l*, then $\int_a^c f(x)\,dx\ (a<c<b)$ also converges at *l*.

Q 4: The improper integral $\int_a^b f(x)\,dx$ converges at *l* and *k* is any constant : show that $\int_a^b k\,f(x)\,dx$ converges at *l* and conversely.

14.4 TESTS FOR CONVERGENCE OF IMPROPER INTEGRALS OF THE FIRST KIND

Let a be the only point of infinite discontinuity of a function f which is integrable on $[a+\in, b]$ all $\in$ satisfying $0<\in<b-a$ and f(x) > 0 for all $x\in[a,b]$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)\,dx$ is that there exists a positive real number k such that

$$\int_{a+c}^b f(x)dx < k \text{ for all f satisfying } 0<\in<b-a$$

Proof : Let $\phi(\in) = \int_{a+c}^b f(x)\,dx, 0<\in<b-a$

Let $0 < \in_1 < \in_2 < b - a$.

Then $\phi(\in_1) - \phi(\in_2) = \int_{a+\in_1}^{b} f(x)\,dx - \int_{a+c^2}^{b} f(x)\,dx = \int_{a+\in_1}^{a+\in_2} f(x)\,dx \geq 0$,

since f(x) > 0 on $[a+\in_1, a+\in_2]$.

$0 < \in_1 < \in_2 \Rightarrow \phi(\in_1) \geq \phi(\in_2)$. This shows that ϕ is a monotone decreasing function on (0, b – a). Therefore $\phi(\in)$ will tend to a finite limit as $\in \to 0+$ if and only if ϕ is bounded above.

In other words, the improper integral $\int_a^b f(x)\,dx$ is convergent if and only if there exists a positive real number k such that

$$\int_{a+c}^{b} f(x)dx < k \text{ for all f satisfying } 0 < \in < b - a$$

Note : If ϕ be not bounded above, then $\phi(\in)$ tends to ∞ as $\in \to 0+$ and the improper integral $\int_a^b f(x)\,dx$ diverges to ∞

Theorem : Comparison test

Let a be the only point of infinite discontinuity of the functions f and g which are both integrable on $[a+\in, b]$ for all $\in$ satisfying $0 < \in < b - a$ and $0 < f(x) \leq kg(x)$ for all $x \in [a, b]$, where k > 0. Then

(i) $\int_a^b g(x)\,dx$ is convergent $\Rightarrow$ $\int_a^b f(x)\,dx$ is convergent

(ii) $\int_a^b f(x)\,dx$ is divergent $\Rightarrow \int_a^b g(x)\,dx$ is divergent

Proof : Since f and g both are integrable on $[a+\in, b]$ and $0 < f(x) \leq kg(x)$ for all $x \in [a+\in, b]$, we have $\int_{a+c}^{b} f(x) = dx \leq k\int_{a+c}^{b} g(x) = dx$.

This holds for all $\in$ satisfying $0 < \in < b - a$

(i) If $\int_a^b g(x) = dx$ be convergent then there exists a positive real number k/ such that

$\int_{a+c}^{b} g(x) = dx < k'$ for all $\in$ satisfying $0 < \in < b - a$

Then $\int_{a+c}^{b} g(x) = dx < kk'$ for $\in$ satisfying $0 < \in < b - a$ and this proves that $\int_a^b f(x)\,dx$ is convergent.

(ii) Let Then

If $\int_a^b f(x)\,dx$ be divergent, then ϕ is not bounded above on (0, b – a) and therefore is not bounded above on (0, b – a).

This proves that $\int_a^b g(x)\,dx$ is divergent.

This completes the proof.

A useful comparison integral :

The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $\mu < 1$

Proof : The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is proper if $\mu \le 0$

Let $\mu > 0$. Let $f(x) = \frac{1}{(x-a)^\mu}$ $a < x \le b$. a is the only point of infinite discontinuity of f. f is integrable on $[a+\in, b]$ for $0 < \in < b-a$ and f (x) > 0 for all $x \in (a, b]$.

Let $\phi(\in) = \int_{a+\in}^b \frac{dx}{(x-a)^\mu}$, $0 < \in < b-a$

If $\mu \ne 1$, $\phi(\in) = \int_{a+\in}^b (x-a)^{-\mu}\,dx = \frac{1}{1-\mu}\left[\frac{1}{(b-a)^{\mu-1}} - \frac{1}{\in^{\mu-1}}\right]$

If $0 < \mu < 1$, $\lim_{\in\to 0+} \phi(\in) = \frac{1}{1-\mu}\,\frac{1}{(b-a)^{\mu-1}}$ and if $\mu > 1$, $\lim_{\in\to 0+} \phi(\in) = \infty$

If $\mu = 1$, $\phi(\in) = \int_{a+\in}^b \frac{dx}{(x-a)} = \log|b-a| - \log|\in|$ and $\lim_{\in\to 0+} \phi(\in) = \infty$

Since $\lim_{\in\to 0+} \phi(\in)$ exists finitely when $0 < \mu < 1$ and $\lim_{\in\to 0+} \phi(\in) = \infty$, when $\mu \ge 1$, the improper integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if an only if $0 < \mu < 1$.

Since the integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is proper if $\mu \le 1$ and the improper integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if an only if $0 < \mu < 1$, it follows that the

integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $\mu < 1$.

Theorem : μ test (A practical test)

Let a be the only point of infinite discontinuity of a function f which is integrable on $[a+\in, b]$ for $0 < \in < b-a$ and f (x) > 0 for all $x \in (a, b]$

If $\lim_{x \to a+} f(x)(x-a)^\mu = 1$ where *l* is a non-zero finite number, then the integral $\int_a^b f(x)\, dx$ is convergent if and only if $\mu < 1$.

Proof : Since $f(x)(x-a)^\mu > 0$ for all $x \in (a, b]$, *l* > 0

Let us choose a positive δ such that $1-\delta > 0$.

Since $\lim_{x \to a+} f(x)(x-a)^\mu = 1$ there exists a point c in (a, b) such that

$$1-\delta < f(x)(x-a)^\mu < 1+\delta \text{ for all } x \in (a, c]$$

or, $$\frac{1-\delta}{(x-a)^\mu} < f(x) < \frac{1+\delta}{(x-a)^\mu} \text{ for all } x \in (a, c]$$

If $\mu < 1$, the integral $\int_a^c \frac{1}{(x-a)^\mu}\, dx$ is convergent.

If $u \geq 1$, the integral $\int_a^c \frac{1}{(x-a)^\mu}\, dx$ is divergent. Since $1-\delta > 0$, if followsfrom the left hand inequality that $\int_a^c f(x)\, dx$ is divergent.

Therefore, $\int_a^c f(x)\, dx$ is convergent if and only if $\mu < 1$ and thereore $\int_a^b f(x)\, dx$ is convergent if and only if $\mu < 1$.

Note : If $\lim_{x \to b-} f(x)(b-x)^\mu = 1$ where l is a non-zero finite number, then the integral $\int_a^b f(x)\, dx$ is convergent if and only if $\mu < 1$.

Example : Show that $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ is convergent if and only if m, n are both positive.

Solution : Let the given integral be $\int_0^1 f(x)dx$. It is a proper integral if $m \geq 1$ and $n \geq 1$. 0 is the only point of infinite discontinuity of f if m < 1 and 1 is the only point of infinite discontinuity of f if n < 1.

Let us examine the convergence $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ when m < 1 and the convergence of $\int_{\frac{1}{2}}^{1} x^{m-1}(1-x)^{n-1}dx$ when n < 1.

Convergence of $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ at 0 when m < 1.

f (x) > 0 for all $x \in \left(0, \frac{1}{2}\right]$. $\lim_{x\to 0+} f(x)x^{1-m} = \lim_{x\to 0+} f(1-x)^{n-1}$ (non-zero finite number.

By μ test $\int_0^{\frac{1}{2}} f(x)\,dx$ is convergent if and only if 1 – m < 1, i,.e. if and only if m > 0.

Convergence of $\int_{\frac{1}{2}}^{1} x^{m-1}(1-x)^{n-1}dx$ at 1 when n < 1.

f (x) > 0 for all $x \in \left(\frac{1}{2}, 1\right]$. $\lim_{x\to 1-} f(x)(1-x)^{1-n} = \lim_{x\to 1-} x^{m-1} = 1$ (non-zero finite number).

By μ test $\int_{\frac{1}{2}}^{1} f(x)dx$ is convergent if and only if 1 – n < 1, i.e. if and only if n > 0.

Therefore both the integrals $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ and $\int_{\frac{1}{2}}^{1} x^{m-1}(1-x)^{n-1}dx$ are convergent if and only if m > 0 and n > 0.

Hence $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if and only if m > 0 and n > 0.

Note : The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, m > 0, n > 0 is called the Beta function and it is denoted by B (m, n)

Theorem : Cauchy

Let a be the only point of infinite discontinuity of a function f which isintegrable on $[a+\in, b]$ for all $\in$ satisfying $0 < \in < b-a$ and f(x) may not keep the same sign on $(a, b]$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)\,dx$ is that for a pre-assigned positive $\in$ there exists a positive $\delta < b-a$ such that

$$\left|\int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx\right| < \in \text{ for all } \lambda_1, \lambda_2 \text{ satisfying } 0 < \lambda_1 < \lambda_2 < \delta.$$

Proof : Let $F(\lambda) = \int_{a+\lambda}^{b} f(x)\,dx,\ 0 < \lambda < b-a$

The improper integral $\int_a^b f(x)\,dx$ is convergent if $\lim_{\lambda \to 0+} F(\lambda)$ exists finitely. By Cauchy's creiterion for the exixtence of finite limits $\lim_{\lambda \to 0+} F(\lambda)$ exists finitely if and only if for a pre-assigned positive $\in$ there corresponds a positive $\delta < b-a$ such that

$$\left|F(\lambda_1 - F(\lambda_2)\right| < \in \text{ for all } \lambda_1, \lambda_2 \text{ satisfying } 0 < \lambda_1 < \lambda_2 < \delta$$

$$\left|\int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx\right| < \in \text{ for all } \lambda_1, \lambda_2 \text{ satisfying } 0 < \lambda_1 < \lambda_2 < \delta$$

This conpletes the proof.

14.5 ABSOLUTE CONVERGENCE

Definition : The improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b |f|(x)\,dx$ be convergent.

Theorem : An absolutely convergent improper integral $\int_a^b |f|(x)\,dx$ (where a is the only point of infinite discontinuity of f in [a, b] and f is integrable on $[a+\in, b]$ for all $\in$ satisfying $0 < \in < b-a$ is convergent.

Proof : Here the integral $\int_a^b |f|(x)\,dx$ is convergent and a is the only point of infinite discontinuity of f in [a, b]

Let $\in > 0$. Then there exists a positive $\delta < b-a$ such that

$$\left|\int_{a+\lambda_1}^{a+\lambda_2} |f|(x)\,dx\right| < \in \text{ for all } \lambda_1, \lambda_2 \text{ satisfying } 0 < \lambda_1 < \lambda_2 < \delta$$

We also have $\left|\int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx\right| \le \int_{a+\lambda_1}^{a+\lambda_2} |f|(x)\,dx$

Therefore $\left|\int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx\right| < \in$ for all λ_1, λ_2 satisfying

$0 < \lambda_1 < \lambda_2 < \delta$

This proves that the integral $\int_a^b f(x)\,dx$ is convergent.

This completes the proof.

Notes : (1) The converse of the theorem is not true. We shall establish this by some examples.

(2) Since $|f(x)|$ is always positive, comparison tests can be applied to establish the convergence of the improper integral $\int_a^b |f|(x)\,dx$

Example : Show that the improper integral $\int_0^1 \frac{\sin \frac{1}{x}}{\sqrt{x}} dx$ is convergent.

Solution : Let $f(x) = \int_0^1 \frac{\sin \frac{1}{x}}{\sqrt{x}} dx,\ x \in (0, 1]$

0 is the only point of infinite discontinuity of . f (x) does not keep the same sign in the interval $(0, 1]$

$\left|\frac{\sin \frac{1}{x}}{\sqrt{x}}\right| \le \frac{1}{\sqrt{x}}$ for all $x \in (0, 1]$ and $\int_0^1 \frac{1}{\sqrt{x}} dx$ is convergent.

Therefore $\int_0^1 \left|\frac{\sin \frac{1}{x}}{\sqrt{x}}\right| dx$ is convergent, i.e. $\int_0^1 \frac{\sin \frac{1}{x}}{\sqrt{x}} dx$ is absolutely convergent and therefore $\int_0^1 \frac{\sin \frac{1}{x}}{\sqrt{x}} dx$ convergent.



CHECK YOU PROGRESS 2

Q 1: Test the convergence of the following integrals :

a) $\int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{\tan x}}$, b) $\int_0^1 \frac{dx}{x^3(1+x)^2}$,

c) $\int_1^3 \frac{dx}{\sqrt{(x-x)}}$

Q 2: Show that if $\int_0^1 \frac{\operatorname{cosec} x}{x^n} dx$ is divergent if $n \geq 1$.

Q 3: Show that $\int_0^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} dx$ is convergent if m > 0, n > 0

14.6 CONVERGENCE OF IMPROPER INTEGRALS OF THE SECOND KIND

Case I. Convergence of the improper integral $\int_a^\infty f(x)dx$ where f is integrable on [a, X] for all X > a.

Let $\phi(X) = \int_a^\infty f(x)dx,\ X > a$.

If $\lim_{X\to\infty} \phi(X)$ exists (finitely) then the improper integral $\int_a^\infty f(x)dx$ is said to be convergent. If the limit be *l*, we write $\int_a^\infty f(x)dx = l$

If $\lim_{X\to\infty} \phi(X)$ does not exist (finitely) then the improper integral $\int_a^\infty f(x)dx$ is said to be divergent.

Example : Let us consider the integral $\int_0^\infty e^{-x} dx$. The integrand is integrable on any closed interval [0, X], X > 0. The integral is improper.

Let $\phi(X) = \int_a^\infty e^{-x} dx,\ X > 0$ Then $\phi(X) = 1 - e^{-x}$.

$\lim_{X\to\infty} \phi(X) = 1$

Therefore the integral $\int_0^\infty e^{-x} dx$ is convergent and $\int_0^\infty e^{-x} dx = 1$.

Example : Let us consider the integral $\int_0^\infty \frac{1}{1+x} dx$. The integrand is integrable on any closed interval [0, X], X > 0. The integral is improper.

Let $\phi(X) = \int_0^\infty \frac{1}{1+x} dx,\ X > 0$. Then $\phi(X) = \log(1+X)$.

$\lim_{X\to\infty} \phi(X) = \infty$. Therefore the integral $\int_0^\infty \frac{1}{1+x} dx$ is divergent.

Case II. Convergence of the improper integral $\int_{-\infty}^b \frac{1}{1+x} dx$ where f is integrable on [X, b] for all X < b.

Let $\phi(X) = \int_X^b f(x)\,dx,\ X < b$

If $\lim_{X \to -\infty}(X)$ exists (finitely) then the improper integral $\int_{-\infty}^b f(x)\,dx$ is said to be convergent. If the limit be *I*, we write $\int_{-\infty}^b f(x)\,dx = I$.

Case III. Convergence of the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ where f is integrable on $[X_1, X_2]$ for all $X_1, X_2 \in R$ satisfying $X_1 < X_2$

Let $c \in R$. If both the integrals $\int_{-\infty}^c f(x)\,dx$ and $\int_c^{\infty} f(x)\,dx$ be convergent according to the definitions I and II above, then the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent and we write.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_x^{\infty} f(x)\,dx$$

Example : Let us consider the integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$. There integrand is integrable on any closed interval $[X_1, X_2]$, $X_2 > X_1$. The integral is improper.

Let us consider the integrals $\int_{-\infty}^a \frac{1}{1+x^2}\,dx$ and $\int_a^{\infty} \frac{1}{1+x^2}\,dx$ where $a \in R$.

Let $\phi(X) = \int_X^a \frac{1}{1+x^2}\,dx,\ X < a$. Then $\phi(X) = \tan^{-1} a - \tan^{-1} X$.

$\lim_{X \to -\infty} \phi(X) = \tan^{-1} a + \frac{\pi}{2}$.

Therefore the improper integral $\int_{-\infty}^a \frac{1}{1+x^2}\,dx$ is convergent.

Let $\psi(X) = \int_a^X \frac{1}{1+x^2}\,dx,\ X > a$. Then $\psi(X) = \tan^{-1} X - \tan^{-1} a$.

$\lim_{X \to -\infty} \psi(X) = \frac{\pi}{2} - \tan^{-1} a$

Therefore the improper integral $\int_a^{\infty} \frac{1}{1+x^2}\,dx$ is convergent.

Consequently, $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ the integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \left(\tan^{-1} a + \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - \tan^{-1} a\right) = \pi.$$

IV. Convergence of the improper integral $\int_{-\infty}^{b} f(x)\,dx$ where f has a finite number of points of infinite discontinuity $c_1, c_2, \ldots, c_m$.

Let $c_1 < c_2 < \ldots < c_m$. If each of the integrals $\int_{-\infty}^{c1} f(x)\,dx$, $\int_{c1}^{c2} f(x)\,dx \ldots \int_{c_{m-1}}^{c_m} f(x)\,dx$ and $\int_{c_m}^{\infty} f(x)\,dx$ be convergent, then the improper integral $\int_{-\infty}^{c1} f(x)\,dx$ is said to be convergent and we write.

$$\int_{-\infty}^{c1} f(x)\,dx = \int_{-\infty}^{c1} f(x)\,dx + \int_{c1}^{c2} f(x)\,dx + \ldots + \int_{c_{m-1}}^{c_m} f(x)\,dx + \int_{c_m}^{\infty} f(x)\,dx$$

14.7 TESTS FOR CONVERGENCE OF IMPROPER INTEGRALS OF THE SECOND KIND

Theorem : Let a function f be integrable on [a, X] for all X > a and f (x) > 0 for all $x \geq a$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^{\infty} f(x)\,dx$ is that there exists a positive real number k such that

$$\int_a^X f(x)\,dx < k \text{ for all } X > a$$

Proof : Let $\phi(X) = \int_a^x f(x)dx,\ X > a$

Let $a < X_1 < X_2$.

Then $\phi(X_2) - \phi(X_1) = \int_a^{x2} f(x)dx - \int_a^{x1} f(x)dx = \int_{x_1}^{x2} f(x)dx > 0$, since f (x) > 0 on $[X_1, X_2]$.

$a < X_1 < X_2 \Rightarrow \phi(X_1) < \phi(X_2)$. This shows that ϕ is monotone increasing function on (a, ∞). Therefore $\phi(X)$ wil tend to a finite limit as $X \to \infty$ if and only if ϕ is bounded above.

In other words, the improper integral $\int_a^{\infty} f(x)dx$ is convergent if and only if there exists a positive real number k such that

$$\int_a^X f(x)\,dx < k \text{ for all } X > a$$

Note : If ϕ be not bounded above, then $\phi(X)$ tends to ∞ as $X \to \infty$ and the improper integral $\int_a^X f(x)\,dx$ diverges to ∞.

Theorem : Comparison test

Let the functions f and g be both integrable on [a, X] for X > a and $0 < f(x) \le kg(x)$ for all $x \ge a$, where k > 0. Then

(i) $\int_a^\infty g(x)\,dx$ is convergent $\Rightarrow \int_a^\infty f(x)\,dx$ is convergent.

(ii) $\int_a^\infty f(x)\,dx$ is divergent $\Rightarrow \int_a^\infty g(x)\,dx$ is divergent.

Proof : Since f and g both are integrable on [a, X] and $0 < f(x) \le kg(x)$ for all $x \in [a, X]$, we have $\int_a^X f(x)\,dx \le k\int_a^X g(x)\,dx$.

(i) If $\int_a^\infty g(x)\,dx$ be convergent then there exists a positive real number k' such that $\int_a^X g(x)\,dx \le k'$ for all X > a.

Then $\int_a^X f(x)\,dx \le kk'$ for all X > a and this proves that $\int_a^\infty f(x)\,dx$ is convergent.

(ii) Let $\int_a^X f(x)\,dx$ be divergent. Then $\lim_{X\to\infty}\int_a^X f(x)\,dx = \infty$.

Since $\int_a^X f(x)\,dx \le \int_a^X g(x)\,dx$ and k > 0 it follows that $\lim_{X\to\infty}\int_a^X g(x)\,dx = \infty$.

This proves that $\int_a^\infty g(x)\,dx$ is divergent.

This completes the proof.

A useful comparison integral :

The integral $\int_a^\infty \frac{dx}{x^\mu}$, where a > 0, is convergent if and only if $\mu > 1$.

Proof : Let $\phi(X) = \int_a^x \frac{dx}{x^\mu}$, $X > a$

If $\mu \ne 1$, we have $\phi(X) = \int_a^x \frac{dx}{x^\mu} = \left[\frac{x^{1-\mu}}{1-\mu}\right]_a^X = \frac{1}{1-\mu}\left[X^{1-\mu} - a^{1-\mu}\right]$

and if $\mu = 1$, $\phi(X) = \int_a^x \frac{dx}{x} = \log X - \log a$

If $\mu = 1$, $\lim_{X\to\infty} \phi(X) = \lim_{X\to\infty} \log X - \log a = \infty$

If $\mu \neq 1$,

$$\lim_{X\to\infty} \phi(X) = \lim_{X\to\infty} \frac{1}{1-\mu}\left[X^{1-\mu} - a^{1-\mu}\right] = \begin{cases} \infty, & \mu < 1 \\ \dfrac{1}{(\mu-1)a^{\mu-1}}, & \mu < 1 \end{cases}$$

Since $\lim_{X\to\infty} \phi(X)$ exists finitely when $\mu > 1$ and $\lim_{X\to\infty} \phi(X) = \infty$ when $\mu \leq 1$ the improper integral $\int_a^\infty \frac{dx}{x^\mu}$ is convergent, if $\mu > 1$ and divergent if $\mu \leq 1$.

This completes the proof.

Theorem : μ **test (A practical test)**

Let f (x) > 0 for all $x \geq a$. If $\lim_{X\to\infty} x^\mu f(x) = l$, where *l* is a non-zero finite number, the improper integral $\int_a^\infty f(x)\,dx$ is convergent if and only if $\mu > 1$.

Proof : Let $\lim_{X\to\infty} x^\mu f(x) = l$. Then *l* > 0 .

Let us choose a positive $\in$ such that *l* – $\in$ > 0. Since $\lim_{X\to\infty} x^\mu f(x) = l$, there exists a positive real number k > a such that

$$\left| x^\mu f(x) - l \right| < \in \text{ for all } x \geq k$$

$$\Rightarrow l - \in < x^\mu f(x) < l + \in \text{ for all } x \geq k$$

$$\Rightarrow \frac{l-\in}{x^\mu} < f(x) < \frac{l+\in}{x^\mu} \text{ for all } x \geq k > a$$

$\int_a^\infty \frac{1}{x^\mu} dx$ is convergent if $\mu > 1$. Since *l* + $\in$ > 0, it follows from the last ineqality that $\int_a^\infty f(x)\,dx$ is convergent if $\mu > 1$.

$\int_a^\infty \frac{1}{x^\mu} dx$ is divergent if $\mu \leq 1$. Since *l* – $\in$ > 0, it follows from the first ineqality that $\int_a^\infty f(x)\,dx$ is divergent if $\mu \leq 1$.

Therefore $\int_a^\infty f(x)\,dx$ is convergent if and only if $\mu > 1$.

Theorem : Cauchy

Let $a \in R$ and a function f be integrable on [a, X] for every X > a.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^\infty f(x)\,dx$ is that for a pre-assigned positive $\in$ there exists a positive number X_0 such that

$$\left|\int_{X_1}^{X_2} f(x)\,dx\right| < \in \text{ for all } X_1, X_2 > X_0.$$

Proof : Let $F(X) = \int_a^x f(x)\,dx, \quad X > a$

This improper integral $\int_a^x f(x)\,dx$ is convergent if $\lim_{X\to\infty} F(X)$ exists finitely.

By Cauchy's criterion for the existence of finite limits $\lim_{X\to\infty} F(X)$ exists finitely if and only if for a pre assigned positive $\in$ there corresponds a positive X_0 such that

$|F(X_1) - F(X_2)| < \in$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$

$\Rightarrow \left|\int_a^{X_1} f(x)\,dx - \int_a^{X_2} f(x)\,dx\right| < \in$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$

$\Rightarrow \left|\int_a^{X_2} f(x)\,dx\right| < \in$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$

This completes the proof.

Absolutely convergent :

Definition : The improper integral $\int_a^\infty f(x)\,dx$ is said to be absolutely convergent if the integral $\int_a^x |f|(x)\,dx$ be convergent .

The improper integral $\int_{-\infty}^\infty f(x)\,dx$ is said to be absolutely convergent if the integral $\int_{-\infty}^\infty |f|(x)\,dx$ be convergent.

Theorem : An absolutely convergent improper integral $\int_a^x f(x)\,dx$ (where f is bounded and integrable on [a, X] for every X > a) is convergent.

Proof : Here f is integrable on [a, X] for every X > a and the improper integral $\int_a^x f(x)\,dx$ is absolutely convergent.

Then the integral $\int_a^x |f|(x)\,dx$ is convergent. Therefore for a pre-assigned positive $\in$ there exists a positive ${}^aX_0$ such that

$$\left|\int_a^{X_2} |f|(x)\,dx\right| < \in \text{ for all } X_1, X_2 \text{ satisfying } X_1, X_2 > X_0$$

We also have $\left|\int_{X_1}^{X_2} f(x)\,dx\right| \le \int_{X_1}^{X_2} |f|(x)\,dx$.

Therefore for a pre-assigned positive $\in$ there exists a positive X_0 such that $\left|\int_{X_1}^{X_2} f(x)\,dx\right| < \in$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$.

This implies that the integral $\int_a^x f(x)\,dx$ is convergent.

This completes the proof.

Notes : (i) The converse of the theorem is not true. We shall establish this by some examples.

(ii) Since | f (x) | is always positive, comparison tests can be applied to establish the convergence of the improper integral $\int_a^x |f|(x)\,dx$.

Example : Examine the convergence of the improper integral $\int_1^\infty f(x)\,dx$, where

$$f(x) = \begin{cases} \dfrac{1}{x^2}, & \text{if } x \text{ be rational} \ge 1 \\ -\dfrac{1}{x^2}, & \text{if } x \text{ be rational} > 1 \end{cases}$$

$|f|(x) = \dfrac{1}{x^2}$, $x \ge 1$. $\int_1^\infty |f|(x)\,dx$ is convergent and therefore $\int_1^\infty f(x)\,dx$ is absolutely convergent.

Consequently, the integral $\int_1^\infty f(x)\,dx$ is convergent.



CHECK YOUR PROGRESS 3

Q 1: Examine the convergence of the improper integral :

a) $\int_0^\infty \dfrac{1}{(1+x)\sqrt{x}}\,dx$ b) $\int_0^\infty \dfrac{1}{x \log x}\,dx$

c) $\int_0^\infty \dfrac{x^{n-1}}{1+x}\,dx$

Q 2: Show that the following improper integrals are absolutely convergent

a) $\int_0^{\infty} \frac{\sin x}{1+x^2}\,dx$ b) $\int_0^{\infty} \frac{\cos x}{1+x^2}\,dx$



14.8 LET US SUM UP

- An improper integral with finite interval of integration [a, b] with its integrand having a finite number of points of infinite discontinuity is said to be Improper integral of the first kind.
- An improper integral with infinite interval of integration and bounded integrand is said to be Improper integral of the second kind.
- An improper integral with infinite interval of integration with its integrand having a finite number of points of infinite discontinuity is said to be Improper integral of the mixed kind.
- For closed and bounded interval [a, b], let the left end point a be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a+\in, b]$ for every $\in$ satisfying $0<\in<b-a$. Let

  $\phi(\in)=\int_{a+\in}^{b} f(x)\,dx, \quad 0<\in<b-a$

  If $\lim_{\in\to 0+}\phi(\in)$ exists (finitely) then the improper integral $\int_a^b f(x)\,dx$ is said to be convergent and we write $\int_a^b f(x)\,dx = I$ where I is the limit.

  If $\lim_{\in\to 0+}\phi(\in)$ does not exist (finitely) then the improper integral $\int_a^b f(x)\,dx$ is a said to be divergent.
- For closed and bounded interval [a, b], let the right end point b of the closed and bounded interval [a, b] be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a, b-\in]$ for every $\in$ satisfying $0<\in<b-a$. Let

  $\phi(\in)=\int_{a}^{b-\in} f(x)\,dx, \quad 0<\in<b-a$

If $\lim_{\in\to 0+}\phi(\in)$ exists (finitely) then the improper integral $\int_a^b f(x)\,dx$ is a said to be convergent. If the limit be *l*, we write $\int_a^b f(x)\,dx = l$.

- Let the end points a, b of the closed and bounded interval [a, b] be the only points of infinite discontinuity of a function f which is bounded and integrable on $[a+\in,\ b-\in']$ for every $\in, \in'$ satisfying $0<\in<b-a$, $0<\in'<b-a$. Let $c\in(a, b)$. If the improper integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ be both convergent according to the definitions given above, then the improper integral $\int_a^b f(x)\,dx$ is said to be convergent and we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

- If the improper integrals

$$\int_a^{c_1} f(x)\,dx,\ \int_{c_1}^{c_2} f(x)\,dx,\ \int_{c_2}^{c_3} f(x)\,dx,\ .....\int_{c_{n-1}}^{c_n} f(x)\,dx,\int_{c_n}^{b} f(x)\,dx$$

all exist such that c_1, c_2, ...c_n are any finite number of points of discontinuity of f in [a, b], where $a \le c_1 \le c_2 < .....< c_{n-1} < c_n \le b$ and

$$\int_a^{c_1} f(x)\,dx = 0 \ \text{ if } c_1 = a \qquad \int_{c_n}^{b} f(x)\,dx = 0 \ \text{ if } b = c_n$$

then we say that $\int_a^b f(x)\,dx$ exists and write

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + ... + \int_{c_n}^{b} f(x)\,dx$$

- A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)\,dx$ is that there exists a positive real number k such that

$\int_{a+\in}^{b} f(x)\,dx < k$ for every f satisfying $0<\in<b-a$.

- Let a be the only point of infinite discontinuity of the functions f and g which are both integrable on $[a, b-\in]$ for all satisfying $0<\in<b-a$ and $0 < f(x) \le kg(x)$ for all $x \in (a, b]$, where k > 0. Then

(i) $\int_a^b g(x)\,dx$ is convergent $\Rightarrow \int_a^b f(x)\,dx$ is convergent

(ii) $\int_a^b f(x)\,dx$ is divergent $\Rightarrow \int_a^b g(x)\,dx$ is divergent

- The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $\mu < 1$
- Let a be the only point of infinite discontinuity of a function f which is integrable on $[a, b-\in]$ for $0 < \in < b-a$ and f (x) > 0 for all $x \in (a, b]$. If $\lim_{x\to a} f(x)(x-a)^\mu = l$ where *l* is a non-zero finite number, then the integral $\int_a^b f(x)\,dx$ is convergent if and only if $\mu < 1$.
- The improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b |f|(x)\,dx$ be convergent.
- An absolutely convergent improper integral $\int_a^b |f|(x)\,dx$ (where a is the only point of infinite discontinuity of f in [a, b] and f is integrable on $[a, b-\in]$ for all $\in$ satisfying $0 < \in < b-a$) is convergent.
- Let a function f be integrable on [a, X] for all X > a and f (x) > 0 for all $x \geq a$. A necessary and sufficient condition for the convergence of the improper integral $\int_a^\infty f(x)\,dx$ is that there exists a positive real number k such that

$$\int_a^X f(x)\,dx < k \text{ for all } X > a$$

- Let the functions f and g be both integrable on [a, X] for all X > a and $0 < f(x) \leq kg(x)$ for all $x \geq a$, where k > 0. Then

(i) $\int_a^\infty g(x)\,dx$ is convergent $\Rightarrow \int_a^\infty f(x)\,dx$ is convergent

(ii) $\int_a^\infty f(x)\,dx$ is divergent $\Rightarrow \int_a^\infty g(x)\,dx$ is divergent.

- The integral $\int_a^\infty \frac{dx}{x^\mu}$, where a > 0, is convergent if and only if $\mu > 1$.
- Let f (x) > 0 for all $x \geq a$. If $\lim_{X\to\infty} x^\mu f(x) = l$, where *l* is a non-zero finite number, the improper integral $\int_a^\infty f(x)\,dx$ is convergent if and

only if $\mu > 1$.

- A necessaryand sufficient condition for the convergence of the improper integral $\int_a^\infty f(x)\,dx$ is that for a pre-assigned positive $\in$ there exists a positive number X_0 such that

  $\left|\int_{X_1}^{X_2} f(x)\,dx\right| < \in$ for all $X_1, X_2 > X_0$.

- The improper integral $\int_a^\infty f(x)\,dx$ is said to be absolutely convergent if the integral $\int_a^\infty |f|(x)\,dx$ be convergent.

- Let improper integral $\int_{-\infty}^\infty f(x)\,dx$ is said to be absolutely convergent if the integral $\int_{-\infty}^\infty |f|(x)\,dx$ be convergent.

- An absolutely convergent improper integral $\int_a^\infty f(x)\,dx$ (where f is bounded and integrable on [a, X] for every X > a) is convergent.

14.9 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1

Ans to Q No 1: Since 0 is a point of infinite discontinuity of the integrand, so it is improper. The integrand is bounded and integrable on $[0+\in, 1]$ for all $\in$ satisfying $0 < \in < 1$. Now,

$$\lim_{\in\to 0+}\int_{0+c}^1 \frac{1}{\sqrt{x}}dx = \lim_{\in\to 0+}\left[2-2\sqrt{\in}\right] = 2$$

Therefore the improper integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ is convergent and $\int_0^1 \frac{1}{\sqrt{x}}dx = 2$.

Ans to Q No 2: Try Yourself.

Ans to Q No 3: Try Yourself.

Ans to Q No 4: Try Yourself.

Check your Progress 2

Ans to Q No 1: Try Yourself.

Ans to Q No 2: Try Yourself.

Ans to Q No 2: Try Yourself.

Check your Progress 3

Ans to Q No 1: Try Yourself.

Ans to Q No 2: Try Yourself.



14.10 FURTHER READING

1) Walter, R. Principles of Mathematical Analysis, 3rd edition , McGraw-Hill, 2017.
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7) Bartle, R.G. Sherbert, D.R. (2013). Introduction to real analysis. Hoboken, NJ : Wiley.
8) Narayan, S., Raisinghania, M.D. Elements of Real Analysis S. Chand & Company Pvt. Ltd. 2015.



14.11 MODEL QUESTIONS

Q 1: Determine the convergence of the improper integrals :

a) $\int_0^2 \frac{1}{x(2-x)}dx$ b) $\int_0^{\pi} \frac{1}{\sin x}dx$

c) $\int_0^1 \log x\, dx$

Q 2: Test the convergene of the following integrals :

a) $\int_0^1 \frac{\log x}{\sqrt{x}}\, dx$ b) $\int_0^1 \frac{\sec x}{x}\, dx$

c) $\int_1^2 \frac{\sqrt{x}}{\log x}\, dx$ d) $\int_1^{\frac{\pi}{2}} \frac{\log x}{x^a}\, dx,\ a<1$

e) $\int_0^1 \frac{x^n}{1-x}\, dx$ f) $\int_0^1 \frac{x^n}{1+x}\, dx$

Q 3: Examine the convergence of the improper integrals :

a) $\int_{e^2}^{\infty} \frac{1}{x \log\log x}\, dx$ b) $\int_1^{\infty} \frac{dx}{x\sqrt{x^2+1}}$

c) $\int_0^{\infty} \frac{\cos x}{1+x^2}dx$

Q 4: Show that the following improper integrals are absolutely convergent

a) $\int_0^{\infty} \frac{\sin x}{\sqrt{x+x^3}}\, dx$ b) $\int_0^{\infty} \frac{x \sin x}{1+x^3}\, dx$

*** ***** ***



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